On the Vassiliev Invariants

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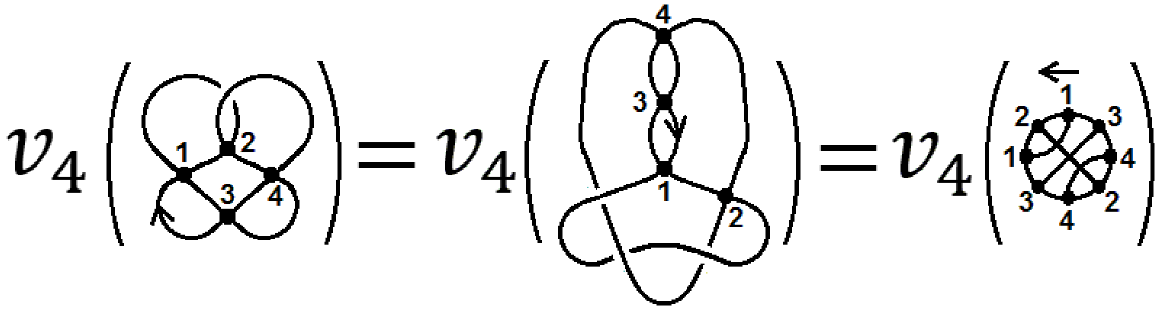
http://evertstenlund.se/knots/



1. Introduction

The most powerful knot invariants devised to date are the Vassiliev Invariants or the Finite Type Invariants [1]. To understand the Vassiliev Invariants the first thing to introduce are virtual knots [2, 3].

2. Virtual knots

Virtual knots are ordinary knots where one or more of the crossings are undesided, i.e. they are neither over- nor undercrossings. Such a crossing is by convention marked in the knot diagram with a dot on top of the crossing. As an example a virtual version of the knot **5**$_2$ with two virtual crossings is shown in figure 1. Here the three remaining crossings are by convention marked with +'s, since they are overcrossings.

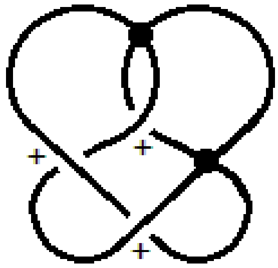


Figure 1: A virtual knot

2. Vassiliev Invariants and virtual knots

The Vassiliev Invariants are defined as a relation between virtual knots. The value of a virtual knot is evaluated as the value obtained as the difference of the values of the two knots where one of the dotted crossings is changed to first an overcrossing and then to an undercrossing. This is illustrated in figure 2, where the knot is affected only in the vicinity of the dotted crossing, leaving the rest of the knot unchanged. The Vassiliev Invariants, v_n, can be evaluated for different degrees, n, where all virtual knots with more than n dotted crossings are given the value 0. For all degrees the unknot is given the value 0. The first nontrivial Vassiliev Invariants, v_2 and v_3, are determined with just one normalization parameter each, whereas higher degrees need more parameters. The number of normalization parameters for degrees 2 – 6 are given in table 1. Values for higher degrees can be found in [4].

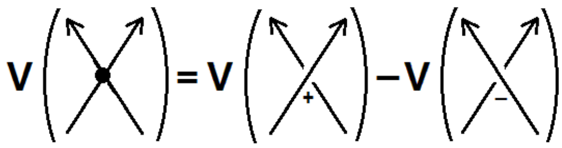


Figure 2: The Vassiliev relation

v_2	v_3	v_4	v_5	v_6
1	1	3	4	9

Table 1: Number of normalisation parameters for different degrees.

3. Chord diagrams

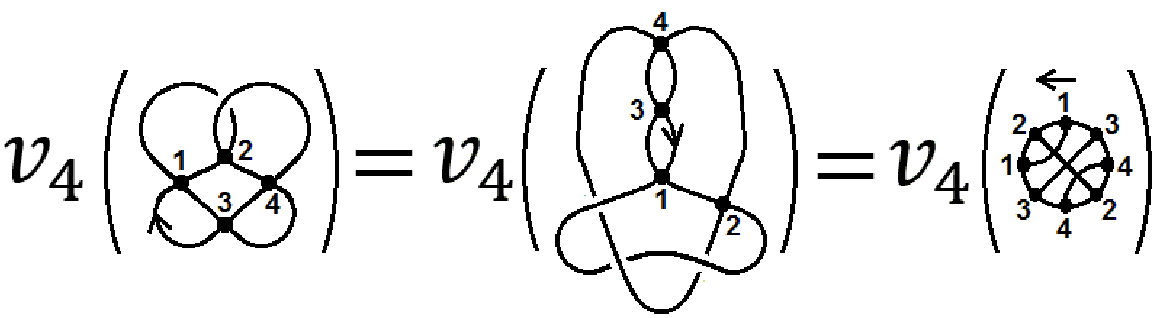


Figure 3: Two knots with the same cycles.

A consequence of the definition of the Vassiliev Invariants is that v_n for two virtual knots having n dotted crossings, where the dotted crossings are passed in exactly the same order, are identical [5]. This is exemplified in figure 3, where virtual versions of the two knots **5₂** and **7₃** are shown. To the right in the figure a circular chord diagram with the same cycle (12134243) is included. This indicates that any virtual knot has a Vassiliev Invariant which equals the one for the corresponding chord diagram.

3. Chord algebra

There exists an algebra for chords, or rather their corresponding Vassiliev Invariants. There are two types of relations which are obeyed by the Vassiliev Invariants, namely the 1-term (1T) relation and the 4-term (4T) relation [5]. Figure 4 illustrates the 1T relation. This type of chord diagrams corresponds to a loops in the corresponding knot. In figure 5 the 4T relation is shown. In both these figures the black parts are the only parts that are modified and that the red parts of the diagrams may contain any number of chords with endpoints at the red part of the circle. All loop free circular chord diagrams with 6 or fewer chords are given in appendix A, together with their relations. Table 2 gives their numbers and how many of them that are independent. The lower row is recognized as the same sequence as in table 1, and the values in the middle row can be found in [6]. Note that the choice of the independent diagrams is somewhat arbitrary.

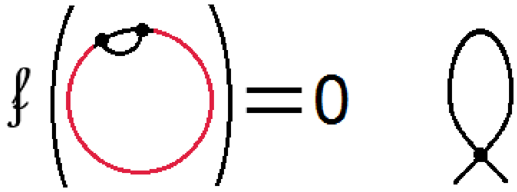


Figure 4: The 1T relation.

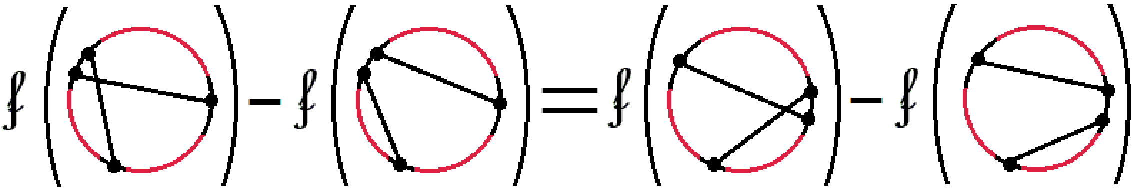


Figure 5: The 4T relation.

2	3	4	5	6
1	2	7	29	196
1	1	3	4	9

Table 2: Number of loop free circular chord diagrams.

4. Chord parametrization of knots

Based on the independent chord diagrams in figure 6 all prime knots of 9 or fewer crossings have been parametrized and are found in appendixes B (3 – 8 crossings) and C (9 crossings). Figure 7 is taken as an example from appendix B and shows the parametrization for knot **7₃**. Note that, in the expression of v_{n+2}, the expression of v_n is included. This is done in order to include all possible normalizations of the invariants. For odd n the upper sign corresponds to the orientation of the knot as given by the diagram, and the lower sign corresponds to its mirror image. The graphs seen to the right of the knot diagram are explained in detail in [7].

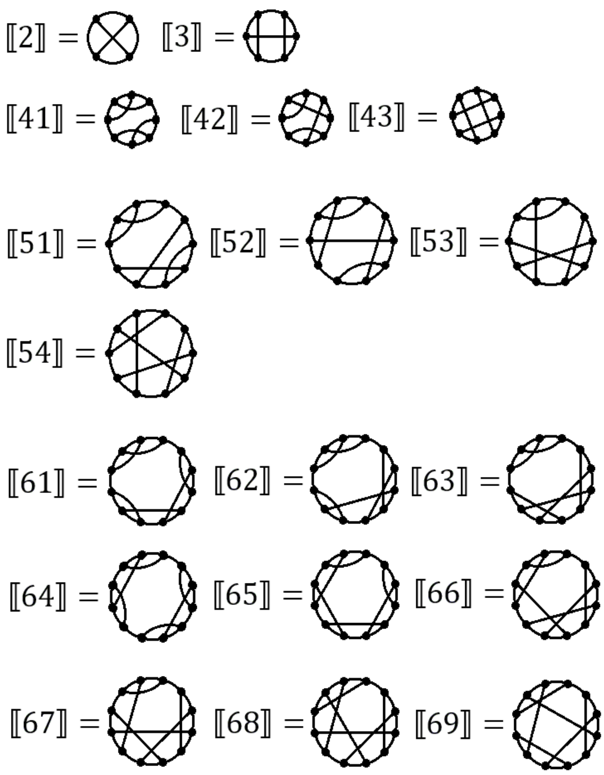
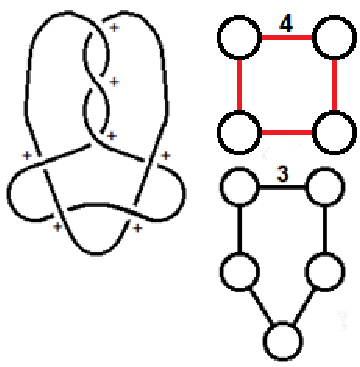


Figure 6: Independent loop free chord diagrams with 6 or fewer chords.

7_3:

$$v_{even} = x_{7.3}$$

$$v_{odd} = \pm y_{7.3}$$



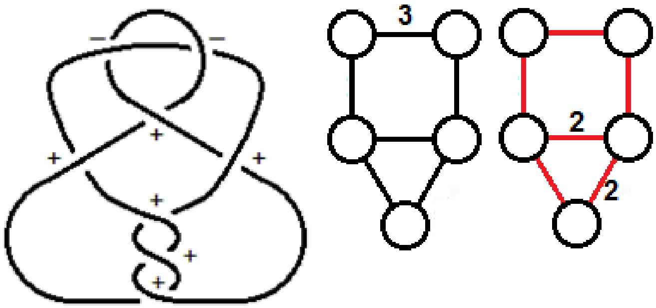
v_2	$5\llbracket 2\rrbracket$
v_3	$\pm 11\llbracket 3\rrbracket$
v_4	$4\llbracket 43\rrbracket + 9\llbracket 42\rrbracket - 7\llbracket 41\rrbracket + 5\llbracket 2\rrbracket$
v_5	$\pm(11\llbracket 54\rrbracket - 8\llbracket 53\rrbracket - 2\llbracket 52\rrbracket + 3\llbracket 51\rrbracket + 11\llbracket 3\rrbracket)$
v_6	$\frac{1}{2}(17\llbracket 69\rrbracket + 31\llbracket 68\rrbracket - 76\llbracket 67\rrbracket + 29\llbracket 66\rrbracket - 58\llbracket 65\rrbracket + 7\llbracket 64\rrbracket + 8\llbracket 63\rrbracket - 2\llbracket 62\rrbracket + 29\llbracket 61\rrbracket + 8\llbracket 43\rrbracket + 18\llbracket 42\rrbracket - 14\llbracket 41\rrbracket + 10\llbracket 2\rrbracket)$

Figure 7: Parametrization of knot **7_3**.

4. Knot parametrization of knots

Due to the duality between virtual knots and chord diagrams, chords diagrams can be parametrized sums of knots. Using this parametrization knots can be parametrized by knots. All prime knots of 9 or fewer crossings have been parametrized in this way and are found in appendixes D (3 – 8 crossings) and E (9 crossings). Figure 8 is taken as an example from appendix D and shows this parametrization for knot $\mathbf{8_{11}}$. Note that in the expression of v_{n+2} the expression of v_n has again been included. This is again done in order to include all possible normalizations of the invariants. In this parametrization they are however obscured and not directly seen in the expressions. For n odd the upper sign corresponds to the orientation of the knot as given by the diagram, and the lower sign corresponds to its mirror image.

$\mathbf{8_{11}}$:



$$v_{even} = x_{8.11}$$

$$v_{odd} = y_{8.11}$$

v_2	$-x_{3.1}$
v_3	$\mp 2y_{3.1}$
v_4	$x_{5.2} - 2x_{5.1} + 6x_{4.1} + 9x_{3.1}$
v_5	$\pm(y_{6.2} + 3y_{6.1} + y_{5.2} - y_{5.1} + 4y_{3.1})$
v_6	$-4x_{7.7} + 2x_{7.6} - 2x_{7.5} + x_{7.3} + 2x_{6.3} - 2x_{6.2} + 3x_{6.1} - 5x_{5.2} + 2x_{5.1} - x_{4.1} + x_{3.1}$

Figure 8: Parametrization of knot $\mathbf{8_{11}}$.

There are altogether 14 prime knots with 7 or fewer crossings but there are only 13 independent chord diagrams with 6 or fewer chords. v_6 for one of these knots can thus be expressed as a sum of the other knots. From the relation

$$5x_{7.7} - 2x_{7.6} + 2x_{7.5} + x_{7.4} - 2x_{7.3} - 2x_{6.3} + 6x_{6.2} - 2x_{6.1} + 4x_{5.2} + x_{5.1} - 10x_{4.1} - 12x_{3.1} = 0$$

the obvious choice to single out seems be knot $\mathbf{7_4}$ since this avoids non-integer coefficients in the expression. Except for this choice the parametrizations of v_2 - v_6 are unique for all knots.

5. Summary

Vassiliev Invariants have been parametrized up to degree 6 in an almost unique way. The invariants can be easily normalized in a convenient way as given in appendix F where each parameter is set to a non-zero value, one at a time, setting all the other parameters to zero.
It should be mentioned that for every knot there exists an integral, the Kontsevich integral, which is a universal Vassiliev Invariant in the meaning that it contain all Vassiliev Invariants of any degree [8]. This complicated subject is however outside the scope of this note.

References

[1] See for instance
Duzhin, Sergei. "Vassiliev Invariant." From *MathWorld*--A Wolfram Web Resource, created by Eric W. Weisstein. http://mathworld.wolfram.com/VassilievInvariant.html
or
https://en.wikipedia.org/wiki/Finite_type_invariant

[2] Kauffman, L., Virtual Knot Theory, Europ. J. Combinatorics (1999) 20, 663-691, arXiv:math.GT/9811028

[3] See for instance
https://en.wikipedia.org/wiki/Virtual_knot

[4] The On-Line Encyclopedia of Integer Sequences® (OEIS®), https://oeis.org/, sequences A007293 and A014596.

[5] Chmutov, S., Duzhin, S., and Mostovoy, J., Introduction to Vassiliev Knot Invariants (2012), University Press, Cambridge. ISBN 978-1-107-02083-2 Hardback.
and references therein.

[6] The On-Line Encyclopedia of Integer Sequences® (OEIS®), https://oeis.org/, sequence A003437.

[7] Stenlund, E., On Knots and Planar Connected Graphs, http://evertstenlund.se/knots/

[8] Kontsevich, M., Vassiliev's knot invariants, , I. M. Gelfand seminar. Part 2, Adv. Soviet Math. 16 137{150, Amer. Math. Soc., Providence, RI, 1993.